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(r*g*)*Regular and (r*g*)*Normal Spaces

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ABSTRACT: The aim of this paper is to introduce the concepts of (r*g*)* regular and (r*g*)*normal spaces and study some of their properties.

KEYWORDS : (r*g*)* closed sets , (r*g*)* closure, (r*g*)* open sets. Mathematics Subject Classification : 54A05

I.INTRODUCTION

Levine [5] in 1963 initiated the study of generalized open sets with the introduction of semi open sets. Then many Topologist introduced different types of open sets like g - open sets, pre - open sets, and semi -open sets etc. and many authors utilized these concepts to define normality and regularity. The concept of g-regular and g-normal and s-regular and s - normal spaces in Topological spaces were introduced and studied by Munshi [12] and Maheswari and Prasad [7]respectively. Arya and Nour [1] obtained some characterization of s- normal spaces. The Authors [8] have already introduced (r*g*)* closed set and investigated some of their properties. In this paper two new classes of spaces called (r*g*)* Regular and (r*g*)* Normal spaces are introduced and some of the basic properties are investigated.

II. PRELIMINARIES

Definition 2.1: A subset A of a space X is called a (r*g*)*closed set [8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is r*g*- open. The complement of (r*g*)*closed set is (r*g*)*open.

Definition 2.2: Let X be a Topological space. Let A be a subset of X. (r*g*)*closure[10] of A is defined as the intersection of all (r*g*)* closed sets containing A.

Definition 2.3: A property P holding good for a topological space(X, \mathfrak{S}) and which is also holds good for every subspace of the topological space is called Hereditary property.

Definition 2.4 : Two non-empty subsets A and B of a topological space are said to be separated iff

$$A \cap cl(B) = \emptyset \text{ and } cl(A) \cap B = \emptyset.$$

Definition 2.5: A Topological space (X, \mathfrak{S}) is said to be (r*g*)*T₀ space[11] if and only if for every pair of distinct points x ,y in X there exists a (r*g*)*open set G such that $x \in G, y \notin G$ or $y \in G$ but $x \notin G$, (i.e) for distinct points x and y of X there exists a (r*g*)*open set G containing one of the points but not the other.

Definition 2.6: A Topological space (X, \mathfrak{S}) is said to be (r*g*)* T₁ space[11] iff for any given pair of distinct points x and y there exists two (r*g*)* open sets G and H such that G contains x but not y and H contains y but not x. The above property is known as (r*g*)*T₁ axiom.

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Definition 2.7: A Topological space (X, \mathfrak{S}) is said to be a $(r^*g^*)T_2$ space [11] or (r^*g^*) Hausdorff space if for every pair of distinct points x_1, x_2 of X there exist disjoint (r^*g^*) open sets U and V of x_1 and x_2 respectively and such that $U \cap V = \emptyset$.

III. (r^*g^*) REGULAR AND $(r^*g^*)T_3$ SPACES

Definition 3.1: A Topological space (X, \mathfrak{S}) is said to be (r^*g^*) regular iff for every closed subset F of X and each point $x \notin F$ there exists two disjoint (r^*g^*) open sets G and H such that $x \in G$ and $F \subset H$.

Definition 3.2: A Topological space is said to be $(r^*g^*)T_3$ space if it is (r^*g^*) regular as well as $(r^*g^*)T_1$ space.

Example 3.3: Let $X = \{a, b, c\}$ $\mathfrak{S} = \{\emptyset, X, \{a\}, \{b, c\}\}$

(r^*g^*) open-sets are $\{\emptyset, X, \{b, c\}, \{a, c\}, \{a, b\}, \{c\}, \{a\}, \{b\}\}$

This space is $(r^*g^*)T_1$ and is (r^*g^*) regular and so is a $(r^*g^*)T_3$ space.

Theorem 3.4: Every subspace of a (r^*g^*) regular space is a (r^*g^*) regular space.

Proof: Let (X, \mathfrak{S}) be a (r^*g^*) regular space and (Y, \mathfrak{S}^*) be its subspace where

$\mathfrak{S}^* = \{Y \cap G : G \text{ is open}\}$ TPT (Y, \mathfrak{S}^*) is also a (r^*g^*) regular space.

Let F^* be closed in Y so that $F^* = Y \cap F$ Where F is closed subset of X .

Let $x \in Y - F^* \Rightarrow x \in Y$ and $x \notin F^*$

$\Rightarrow x \notin Y \cap F \Rightarrow x \notin F$

Since (X, \mathfrak{S}) is (r^*g^*) regular and F is closed, there exist (r^*g^*) open sets G and H such that $x \in G$ and $F \subset H$ and such that $G \cap H = \emptyset$. Now $F \subset G \Rightarrow Y \cap F \subset Y \cap G \Rightarrow F^* \subset G^*$

$x \in G \Rightarrow x \in Y \cap G = G^*$

Also $(Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H) = Y \cap \emptyset = \emptyset \Rightarrow G^* \cap H^* = \emptyset$

For $x \in Y - F^* \& G^*$ there exists (r^*g^*) open sets G^* and H^* such that $x \in H^*$, $F^* \subset G^*$ and such that

$G^* \cap H^* = \emptyset$. Hence (Y, \mathfrak{S}^*) is (r^*g^*) Regular.

Remark 3.5: Since $(r^*g^*)T_1$ satisfies hereditary property, $(r^*g^*)T_3$ also satisfies hereditary property (or) subspace of a $(r^*g^*)T_3$ space is a $(r^*g^*)T_3$ space.

Theorem 3.6: Every $(r^*g^*)T_3$ space is a $(r^*g^*)T_2$ space.

Proof: Since (X, \mathfrak{S}) is $(r^*g^*)T_3$ it is both $(r^*g^*)T_1$ and (r^*g^*) regular. Also (X, \mathfrak{S}) is $(r^*g^*)T_1 \Rightarrow$ every singleton subset $\{x\}$ of X is a (r^*g^*) closed. Let $\{x\}$ be a (r^*g^*) closed subset of X and $y \in X - \{x\}$. Then we have $x \neq y$ since (X, \mathfrak{S}) is (r^*g^*) regular there exists two (r^*g^*) open sets G and H such that $\{x\} \subset G$, $y \in H$, and such that $G \cap H = \emptyset$ (or) G and H are disjoint (r^*g^*) open sets containing x and y respectively. Since x and y are arbitrary, For Every pair of distinct points there exist disjoint (r^*g^*) open sets. Hence (X, \mathfrak{S}) is $(r^*g^*)T_2$.

Theorem 3.7: Every (r^*g^*) compact Hausdorff space is a $(r^*g^*)T_3$ space and hence a (r^*g^*) Regular space.

Proof: Let (X, \mathfrak{S}) be a compact Hausdorff space, that is a $(r^*g^*)T_2$ space. But every

$(r^*g^*)T_2$ is $(r^*g^*)T_1$. To prove that it is $(r^*g^*)T_3$ space it is sufficient to prove that it is (r^*g^*) Regular. Let F be a closed subset of X , and $x \notin F$. Now $x \in X - F$ so that any point $y \in F$ is a point of X which is different from x .

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Since (X, \mathfrak{S}) is $(r^*g^*)T_2$ space corresponding to x and y there exists two (r^*g^*) open sets H_y and G_y such that $G_y \cap H_y = \emptyset$ where $x \in H_y$ and $y \in G_y$. Now let \mathfrak{S}^* denote the relative Topology for F so that the collection $C' = \{ F \cap H_y : y \in F \}$ is a $(r^*g^*)\mathfrak{S}^*$ open cover for F . But F is closed and since (X, \mathfrak{S}) is (r^*g^*) compact (F, \mathfrak{S}^*) is also (r^*g^*) compact. Hence a finite subcover off (or) there exists points $y_1 \dots y_n$ in F such that $C' = \{ F \cap H_{y_i} / i = 1 \dots n \}$ is a finite sub cover for F . Now $F = \cup \{ F \cap H_{y_i} : i = 1 \dots n \}$ or $F = F \cap \{ \cup \{ H_{y_i} / i = 1 \dots n \} \}$
Hence $F \subset \cup \{ H_{y_i} / i = 1 \dots n \}$ or $F \subset H$ where $H = \cup \{ H_{y_i} / i = 1 \dots n \}$ is (r^*g^*) open set containing H , being the Union of (r^*g^*) open sets. Again G_{y_i} for $i = 1, 2, \dots n$ is (r^*g^*) open set containing x and hence $G = \cap \{ G_{y_i} / 1 \dots n \}$ is also a (r^*g^*) open set Containing x .

Also $G \cap H = \emptyset$, otherwise $G_{y_i} \cap H_{y_i} \neq \emptyset$ for some i . Hence corresponding to each closed F and an element x in $X - F$ we have two (r^*g^*) open sets G and H such that $x \in G$, $F \subset H$ such that $G \cap H = \emptyset$. Hence (X, \mathfrak{S}) is (r^*g^*) Regular. Since it is $(r^*g^*)T_2$ it is $(r^*g^*)T_1$ and hence (X, \mathfrak{S}) is $(r^*g^*)T_3$.

IV. (r^*g^*) NORMAL AND $(r^*g^*)T_4$ SPACES

Definition 4.1: A Topological space (X, \mathfrak{S}) is said to be (r^*g^*) Normal iff corresponding to every pair A and B of disjoint closed sets of X there exists (r^*g^*) opensets G and H such that $A \subset G$, $B \subset H$ and $G \cap H = \emptyset$.

Definition 4.2: A Topological space (X, \mathfrak{S}) is said to be $(r^*g^*)T_4$ Space if it is (r^*g^*) Normal as well as $(r^*g^*)T_1$.

Example 4.3: In example 3.3 the space (X, \mathfrak{S}) is (r^*g^*) Normal.

Theorem 4.4: Every closed subspace of a (r^*g^*) Normal space is (r^*g^*) Normal.

Proof: Let (X, \mathfrak{S}) be (r^*g^*) Normal and Y be a closed subspace of X . TPT (Y, \mathfrak{S}^*) is (r^*g^*) Normal where \mathfrak{S}^* is the relative topology.

Let A^* and B^* be two closed disjoint subsets of Y . Then we have $A^* = Y \cap A$

$B^* = Y \cap B$ where A and B are closed sets in X . Now Y is closed and A and B are closed.

Hence $Y \cap A$ and $Y \cap B$ are closed which are disjoint subsets of X . Hence A^* and B^* are closed subsets of X which are disjoint. Since (X, \mathfrak{S}) is (r^*g^*) Normal, Corresponding to the disjoint closed subsets A^* and B^* of X there exists (r^*g^*) open subsets G and H such that

$A^* \subset G$, $B^* \subset H$, $G \cap H = \emptyset$. Now $A^* \subset G \Rightarrow A^* \subset Y \cap G$ (since $A^* \subset Y$)

$B^* \subset H \Rightarrow B^* \subset Y \cap H$ (since $B^* \subset Y$)

Also $G \cap H = \emptyset \Rightarrow (Y \cap G) \cap (Y \cap H) = \emptyset$. Also G and H are (r^*g^*) open and hence $Y \cap G$ and $Y \cap H$ are (r^*g^*) open in Y . Now Corresponding to two closed sets A^* and B^* of Y there exist two \mathfrak{S}^* open sets $Y \cap G$ and $Y \cap H$ such that $A^* \subset Y \cap G$ $B^* \subset Y \cap H$ such that $(Y \cap H) \cap (Y \cap G) = \emptyset$. Hence (Y, \mathfrak{S}^*) is (r^*g^*) Normal.

Theorem 4.5: Every subspace of a $(r^*g^*)T_4$ space is a $(r^*g^*)T_4$ space.

Remark 4.6: From the above theorem $(r^*g^*)T_4$ satisfies Hereditary Property.

Theorem 4.7: Every $(r^*g^*)T_4$ space is $(r^*g^*)T_3$ space.

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Let (X, \mathfrak{S}) be $(r^*g^*)T_4$. Then it is $(r^*g^*)T_1$ and (r^*g^*) Normal space. If we prove that it is (r^*g^*) regular then it will be $(r^*g^*)T_3$.

Let F be any closed subset of X and $x \notin F$ that is $x \in X - F$ so that $\{x\}$ is a (r^*g^*) closed set as (X, \mathfrak{S}) is $(r^*g^*)T_1$. So consider $\{x\}$ and F since (X, \mathfrak{S}) is (r^*g^*) Normal, there exists two (r^*g^*) open sets G and H and such that $\{x\} \subset G, F \subset H, G \cap H = \emptyset$. Where $x \in G$ and $F \subset H$ & F is (r^*g^*) closed. Hence (X, \mathfrak{S}) is (r^*g^*) regular and it being $(r^*g^*)T_1$ it is $(r^*g^*)T_3$.

Theorem 4.8: Every (r^*g^*) Normal space is not necessarily a $(r^*g^*)T_1$.

This can be seen from the following example.

Example 4.9: Let $X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. (r^*g^*) closed sets are $\emptyset, X, \{c\}, \{b, c\}, \{a, c\}$. (r^*g^*) open sets are $\emptyset, X, \{a\}, \{b\}, \{a, b\}$. Here (X, τ) is (r^*g^*) normal. Consider b and $\{c\}$. There are no disjoint (r^*g^*) open sets containing b and $\{c\}$. Therefore (X, \mathfrak{S}) is not (r^*g^*) regular and is not $(r^*g^*)T_1$ and is not $(r^*g^*)T_4$.

Theorem 4.10: Every (r^*g^*) compact Hausdorff space is $(r^*g^*)T_4$.

Theorem 4.11: A Topological space (X, \mathfrak{S}) is (r^*g^*) normal iff for any closed set F and open set G containing F there exist an (r^*g^*) open set V such $F \subset V$ and $(r^*g^*)cl(V) \subset G$

Proof: Let (X, \mathfrak{S}) be (r^*g^*) Normal and let F be a closed set which is $\subset G$

Hence $X - G$ is Closed set which is disjoint from F . Since (X, \mathfrak{S}) is (r^*g^*) normal Corresponding to $X - G$ and F there exists two (r^*g^*) open sets H and K such that

$$X - G \subset H, F \subset K \text{ and } H \cap K = \emptyset \dots \dots \dots (1)$$

$$\text{Now } X - G \subset H \Rightarrow X - H \subset G \dots \dots \dots (2)$$

$$\text{Also } H \cap K = \emptyset \Rightarrow K \subset X - H$$

$$\Rightarrow (r^*g^*)cl(K) \subset X - H \text{ [Since } X - H \text{ is closed } (r^*g^*)cl(X - H) = X - H]$$

$$\Rightarrow (r^*g^*)cl(K) \subset (X - H)$$

$$\Rightarrow (r^*g^*)cl(K) \subset G \dots \dots \dots (3)$$

By (1) & (3) K is a (r^*g^*) open set such that $F \subset K$ & $(r^*g^*)cl(K) \subset G$

Conversely given any closed set F and a open G containing F there exists (r^*g^*) open set

$F \subset V, (r^*g^*)cl(V) \subset G$. TPT (X, \mathfrak{S}) is (r^*g^*) Normal

Let A and B be two disjoint closed subsets of X . Since A and B are disjoint.

$$A \cap B = \emptyset \Rightarrow A \subset X - B \text{ Thus } A \text{ is closed Subset which is } \subset X - B$$

$$\text{Hence } A \subset V \text{ and } (r^*g^*)cl(V) \subset X - B$$

$$(r^*g^*)cl(V) \subset X - B \Rightarrow B \subset X - (r^*g^*)cl(V)$$

$$\text{Also } (X - (r^*g^*)cl(V)) \cap V \subset (X - V) \cap V = \emptyset$$

$$(X - (r^*g^*)cl(V)) \cap V = \emptyset$$

Since $(r^*g^*)cl(V)$ is (r^*g^*) closed $X - (r^*g^*)cl(V)$ is open.

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Corresponding to A and B there exists (r^*g^*) -open sets there exist V and X - (r^*g^*) -cl(V) such that $A \subset V$, $B \subset X - (r^*g^*)$ -cl(V) & $(X - (r^*g^*)$ -cl(V)) \cap V = ϕ , Hence (X, \mathfrak{T}) is (r^*g^*) -Normal.

V. COMPLETELY NORMAL AND (r^*g^*) -T₅ SPACES

Definition 5.1: A Topological space (X, \mathfrak{T}) is said to be completely (r^*g^*) -Normal iff for any two separated sets A and B of X there exist (r^*g^*) -open sets G and H such that $A \subset G$, $B \subset H$ and $G \cap H = \phi$.

Definition 5.2: A Topological space (X, \mathfrak{T}) is said to be (r^*g^*) -T₅ Space if it is completely (r^*g^*) -Normal as well as (r^*g^*) -T₁.

Example 5.3: Let $X = \{a, b, c\}$ $\mathfrak{T} = \{ \phi, X, \{a\}, \{b,c\} \}$

(r^*g^*) -open sets are $\{ \phi, X, \{b,c\}, \{a,c\}, \{a,b\}, \{c\}, \{a\}, \{b\} \}$

(r^*g^*) -closed sets are $\{ \phi, X, \{b,c\}, \{a,c\}, \{a,b\}, \{c\}, \{a\}, \{b\} \}$

Here (X, \mathfrak{T}) (r^*g^*) -T₅ is completely normal and (r^*g^*) -T₁ and (r^*g^*) -T₅.

Theorem 5.4: Every completely (r^*g^*) -Normal space is (r^*g^*) -Normal and hence every (r^*g^*) -T₅ is (r^*g^*) -T₄.

Proof: Let A and B be two disjoint closed subsets of X. $\therefore A = cl(A)$ and $B = cl(B)$ and $A \cap B = \phi$

Which implies $cl(A) \cap B = A \cap B = \phi$ and $A \cap cl(B) = A \cap B = \phi$

This implies A & B are separated sets. Since (X, \mathfrak{T}) is (r^*g^*) -completely Normal there exist two disjoint (r^*g^*) -open sets G and H such that $A \subset G$, $B \subset H$ and $G \cap H = \phi$. Hence (X, \mathfrak{T}) is (r^*g^*) -Normal. Now (X, \mathfrak{T}) is (r^*g^*) -completely Normal implies it is (r^*g^*) -T₁. Hence (X, \mathfrak{T}) is (r^*g^*) -Normal and (r^*g^*) -T₁. In other words (X, \mathfrak{T}) is a (r^*g^*) -T₄ space.

Theorem 5.5: Every subspace of a (r^*g^*) -completely Normal space is a (r^*g^*) -completely Normal space.

Proof: Let (X, \mathfrak{T}) be (r^*g^*) -completely Normal and Y be a closed subspace of X.

TPT (Y, \mathfrak{T}^*) is completely (r^*g^*) -Normal where \mathfrak{T}^* is the relative topology.

Let A and B be two separated sets of Y. Then we have $cl(A^*) \cap B = \phi$

$A \cap cl(B^*) = \phi$. Now $cl(A^*) = cl(A) \cap Y$ $cl(B^*) = cl(B) \cap Y$

Now $\phi = A \cap cl(B^*) = A \cap (cl(B) \cap Y)$

$$= (A \cap cl(B)) \cap Y$$

$$= (A \cap cl(B)) \quad \text{Since } A \cap cl(B) \subset A \subset Y$$

Similarly we have $\phi = cl(A^*) \cap B = (cl(A) \cap Y) \cap B$

$$= (cl(A) \cap B) \cap Y$$

$= cl(A) \cap B$ [Since $cl(A) \cap B \subset B \subset Y$]

Which implies A and B are separated sets of X and since X is (r^*g^*) -completely Normal there exists (r^*g^*) -open sets G and H such that $A \subset G$, $B \subset H$, $G \cap H = \phi$

Now $A \subset G$, and also $A \subset Y$, implies $A \subset G \cap Y$ implies $A \subset G^*$

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$B \subset H, B \subset Y \implies B \subset H \cap Y \implies B \subset H^*$ where are open sets of Y . Also

$G^* \cap H^* = (G \cap Y) \cap (H \cap Y) = (G \cap H) \cap Y = \emptyset$.

Hence (Y, \mathfrak{T}^*) is $(r^*g^*)^*$ completely Normal.

Remark 5.6: The Space T_5 satisfies Hereditary property.

Proof: Follows from the above theorem and the fact that every subspace of $(r^*g^*)^*T_1$ is a $(r^*g^*)^*T_1$

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